Part I: (60 Points/12 Points each) Problems 1-6: Evaluate the definite integrals and find the indefinite integrals. Be sure to write down your evil plan(s) or strategies; especially if you get stuck on a problem. Provide exact answers only. No graphing calculator is permitted. No table formulas may be used. **I'M ONLY GRADING 5 OUT OF THE 6 PROBLEMS, SO PLEASE CROSS OUT THE PROBLEM YOU DO NOT WANT ME TO GRADE.**

1. $\int \tan^3 4x \sec 4x\,dx$

$= \int (\tan^2 4x)(\tan 4x \sec 4x\,dx)$

$= \int (\sec^2 4x - 1)(\tan 4x \sec 4x\,dx)$

$= \frac{1}{4}\int (\sec 4x)^2 (4\sec 4x \tan 4x\,dx) - \frac{1}{4}\int 4\sec 4x \tan 4x\,dx$

$= \frac{1}{4}\frac{\sec^3 4x}{3} - \frac{1}{4}\sec 4x + C$

$= \boxed{\frac{1}{12}\sec^3 4x - \frac{1}{4}\sec 4x + C}$

$$\int u\,dv = uv - \int v\,du$$

2. $\int \arctan x\,dx = (\arctan x)(x) - \int (x)\frac{dx}{1+x^2}$

$$= x\arctan x - \frac{1}{2}\int x(1+x^2)^{-1}\,dx\,(2)$$

$$= x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

$$= \boxed{x\arctan x - \ln\sqrt{1+x^2} + C}$$

Evil Plan

IBP

$u = \arctan x$

$du = \dfrac{dx}{1+x^2}$

$dv = dx$

$v = x$

$g(x) = 1+x^2$

$g'(x) = 2x$

3. $\int_0^4 \frac{x}{\sqrt{6x+1}}dx = \int_0^4 x(6x+1)^{-1/2}\,dx$

Evil Plan

IBP or u-sub and write x in terms of u.

$u = x$

$du = dx$

$dv = (6x+1)^{-1/2}\,dx$

$v = \frac{1}{6}\cdot\frac{(6x+1)^{1/2}}{\frac{1}{2}}$

$v = \frac{1}{3}(6x+1)^{1/2}$

$$\int x(6x+1)^{-1/2}dx = (x)\left[\tfrac{1}{3}(6x+1)^{1/2}\right] - \int \tfrac{1}{3}(6x+1)^{1/2}dx$$

$$= \frac{x\sqrt{6x+1}}{3} - \frac{1}{3}\cdot\frac{1}{6}\int 6(6x+1)^{1/2}dx$$

$$= \frac{x\sqrt{6x+1}}{3} - \frac{1}{18}\cdot\frac{(6x+1)^{3/2}}{3/2} + C$$

$$= \frac{x\sqrt{6x+1}}{3} - \frac{1}{27}(6x+1)^{3/2} + C$$

So...

$$\int_0^4 \frac{x}{\sqrt{6x+1}}dx = \left[\frac{x\sqrt{6x+1}}{3} - \frac{(6x+1)^{3/2}}{27}\right]_{x=0}^{x=4}$$

$$= \left[\frac{4\sqrt{25}}{3} - \frac{(\sqrt{25})^3}{27}\right] - \left[0 - \frac{(\sqrt{1})^3}{27}\right]$$

$$= \frac{20}{3} - \frac{125}{27} + \frac{1}{27}$$

$$= \frac{180-124}{27}$$

$$= \boxed{\frac{56}{27}}$$

4. $\displaystyle\int \frac{x^2}{\sqrt{36-x^2}}\,dx = \int \frac{(6\sin\theta)^2}{\cancel{6\cos\theta}} \cdot \cancel{6\cos\theta\, d\theta}$

$$= 36\int \sin^2\theta \, d\theta$$

$$= 36\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= 18\int (1-\cos 2\theta)\, d\theta$$

$$= 18\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= 18\left(\theta - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C$$

$$= 18\left[\arcsin\frac{x}{6} - \left(\frac{x}{6}\right)\left(\frac{\sqrt{36-x^2}}{6}\right)\right] + C$$

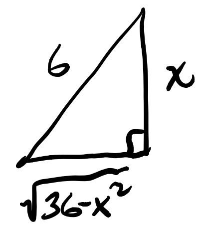
$$= \boxed{18\arcsin\left(\frac{x}{6}\right) - \frac{1}{2}x\sqrt{36-x^2} + C}$$

Evil Plan

Trig sub

$x = 6\sin\theta$

$\frac{x}{6} = \sin\theta$



$x = 6\sin\theta$

$\frac{dx}{d\theta} = 6\cos\theta$

$dx = 6\cos\theta\, d\theta$

$\sqrt{36-x^2} = \sqrt{36-36\sin^2\theta}$

$= 6\sqrt{1-\sin^2\theta}$

$= 6\cos\theta$

$x = 6\sin\theta$

$\theta = \arcsin\frac{x}{6}$

$\sin 2\theta = 2\sin\theta\cos\theta$

5. $\int \cos^2 \theta \sin 3\theta d\theta = \int \frac{1+\cos 2\theta}{2} \cdot \sin 3\theta \, d\theta$

$$= \frac{1}{2} \int (1+\cos 2\theta) \sin 3\theta \, d\theta$$

$$= \frac{1}{2} \left[\int \sin 3\theta \, d\theta + \int \sin 3\theta \cos 2\theta \, d\theta \right]$$

$$= \frac{1}{2} \left[-\frac{\cos 3\theta}{3} + \frac{1}{2} \int (\sin(3-2)\theta + \sin(3+2)\theta) \, d\theta \right]$$

$$= -\frac{1}{6} \cos 3\theta + \frac{1}{4} \int (\sin\theta + \sin 5\theta) \, d\theta$$

$$= -\frac{1}{6} \cos 3\theta + \frac{1}{4} \left(-\cos\theta - \frac{\cos 5\theta}{5} \right) + C$$

$$= \boxed{-\frac{1}{6} \cos 3\theta - \frac{1}{4} \cos\theta - \frac{1}{20} \cos 5\theta + C}$$

6. $\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx$

$= \int \left[2x + \frac{x+5}{x^2-2x-8}\right] dx$

$= x^2 + \int \left(\frac{3/2}{x-4} - \frac{1/2}{x+2}\right) dx$

$= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$

$= x^2 + \ln(|x-4|)^{3/2} - \ln(|x+2|)^{1/2} + C$

$= x^2 + \ln\left[\frac{|x-4|^{3/2}}{|x+2|^{1/2}}\right] + C$

$= \boxed{x^2 + \ln\sqrt{\left|\frac{(x-4)^3}{x+2}\right|} + C}$

Evil Plan

1) Long Division

$$\begin{array}{r} 2x + \frac{x+5}{x^2-2x-8} \\ (x^2-2x-8)\overline{)\,2x^3-4x^2-15x+5} \\ -(2x^3-4x^2-16x) \\ \hline x+5 \end{array}$$

2) PFD

$\frac{x+5}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$

$x+5 = A(x+2) + B(x-4)$

$x+5 = Ax + 2A + Bx - 4B$

$1x + 5 = (A+B)x + (2A-4B)$

$A+B = 1 \rightarrow A = 1-B = 1-(-\frac{1}{2}) = \frac{3}{2}$

$2A-4B = 5$

$\hookrightarrow 2(1-B)-4B = 5$

$2-2B-4B = 5$

$-6B = 3$

$B = -1/2$

Part II: (12 Points) Problems 7-8: Find the indefinite integrals. Be sure to write down your evil plan(s) or strategies; especially if you get stuck on a problem. Provide exact answers only. **I'M ONLY GRADING 1 OUT OF THE 2 PROBLEMS, SO PLEASE CROSS OUT THE PROBLEM YOU DO NOT WANT ME TO GRADE.**

7. $\int e^{-x} \sin 2x dx = (\sin 2x)(-e^{-x}) - \int (-e^{-x})(2\cos 2x\,dx)$

IBP twice and equate like integrals

$u_1 = \sin 2x$

$du_1 = 2\cos 2x\,dx$

$dv_1 = e^{-x}dx$

$v_1 = -e^{-x}$

$u_2 = \cos 2x$

$du_2 = -2\sin 2x\,dx$

$dv_2 = e^{-x}$

$v_2 = -e^{-x}$

$$\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\,dx$$

$$\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x + 2\left[(\cos 2x)(-e^{-x}) - \int(-e^{-x})(-2\sin 2x\,dx)\right]$$

$$\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x - 2e^{-x}\cos 2x - \cancel{4\int e^{-x}\sin 2x\,dx}$$

$$+4\int e^{-x}\sin 2x\,dx \qquad \cancel{+4\int e^{-x}\sin 2x\,dx}$$

$$5\int e^{-x}\sin 2x\,dx = -e^{-x}(\sin 2x + 2\cos 2x)$$

$$\boxed{\int e^{-x}\sin 2x\,dx = -\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x) + C}$$

8. $\int \sec^3 4x\,dx = \int \sec^2 4x \sec 4x\,dx$

IBP

$u = \sec 4x$

$du = 4\sec 4x \tan 4x\,dx$

$dv = \sec^2 4x\,dx$

$v = \frac{1}{4}\tan 4x$

$\int \sec^3 4x\,dx = (\sec 4x)(\tfrac{1}{4}\tan 4x) - \int (\tfrac{1}{4}\tan 4x)(4\sec 4x \tan 4x\,dx)$

$\int \sec^3 4x\,dx = \frac{1}{4}\sec 4x \tan 4x - \int \tan^2 4x \sec 4x\,dx$

$\int \sec^3 4x\,dx = \frac{1}{4}\sec 4x \tan 4x - \int (\sec^2 4x - 1)\sec 4x\,dx$

$\int \sec^3 4x\,dx = \frac{1}{4}\sec 4x \tan 4x - \cancel{\int \sec^3 4x\,dx} + \int \sec 4x\,dx$

$+\int \sec^3 4x\,dx \qquad\qquad + \cancel{\int \sec^3 4x\,dx}$

$2\int \sec^3 4x\,dx = \frac{1}{4}\sec 4x \tan 4x + \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$

$\boxed{\int \sec^3 4x\,dx = \frac{1}{8}\left(\sec 4x \tan 4x + \ln|\sec 4x + \tan 4x|\right) + C}$

Part III: (18 Points). Problems 9-10. Evaluate the following limits. Exact answers only, please.

9. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ $\overset{D.S.}{=} 1^\infty$ LR doesn't apply

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x}$

$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$

$\frac{0}{0}$ so LR applies

$= \lim_{x\to\infty} e^{\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}}$

$\overset{LR}{=} e^{\lim_{x\to\infty}\frac{-1/x^2}{1+1/x} \div \left(-\frac{1}{x^2}\right)}$

$= e^{\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}} \rightarrow \overset{D.S.}{=} e^1 \rightarrow = \boxed{e}$

10. $\lim_{x\to1}\frac{\ln x}{\sin \pi x}$ $\overset{D.S}{=} \frac{0}{0}$ so LR applies

$\lim_{x\to1}\frac{\ln x}{\sin\pi x} \overset{LR}{=} \lim_{x\to1}\frac{\frac{1}{x}}{\pi\cos\pi x}$

$\overset{D.S.}{=} \frac{1}{\pi(-1)}$

$= \boxed{-\frac{1}{\pi}}$

Part IV: (10 Points). Problem 11. Solve the following application. Exact answers only, please.

Find the area of the region bounded by $f(x)=\cos^4 x$, $y=0$, $x=0$, and. $x=\frac{\pi}{3}$.

$$A = \int_0^{\pi/3} \cos^4 x \, dx$$

$$A = \int_0^{\pi/3} (\cos^2 x)^2 \, dx$$

$$A = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$A = \frac{1}{4}\int_0^{\pi/3} (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$A = \frac{1}{4}\int_0^{\pi/3} \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$A = \frac{1}{4}\left[x + \frac{\cancel{2}\sin 2x}{\cancel{2}} + \frac{1}{2}x + \frac{1}{2}\frac{\sin 4x}{4}\right]_{x=0}^{x=\pi/3}$$

$$A = \left(\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{8}\sin 4x\right|_{x=0}^{x=\pi/3}$$

$$A = \left[\left(\frac{3}{8}\cdot\frac{\pi}{3} + \frac{1}{4}\sin(2\pi/3) + \frac{1}{8}\sin(4\pi/3)\right) - (0+0+0)\right)$$

$$A = \frac{\pi}{8} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{16}$$

$$\boxed{A = \frac{1}{16}(2\pi + \sqrt{3}) \text{ sq. units}}$$

EXTRA CREDIT: UP TO 12 POINTS

$\int x^5 e^{x^2} dx = \int x^4 x e^{x^2} dx$

$u_1 = x^4$

$du_1 = 4x^3 dx$

$dv_1 = \frac{2}{2} x e^{x^2} dx$

$v_1 = \frac{1}{2} e^{x^2}$

$u_2 = x^2$

$du_2 = 2x dx$

$dv_2 = 2x e^{x^2} dx$

$v_2 = e^{x^2}$

$= (x^4)(\frac{1}{2}e^{x^2}) - \int (\frac{1}{2}e^{x^2})(4x^3 dx)$

$= \frac{1}{2}x^4 e^{x^2} - \int x^2 \cdot 2x e^{x^2} dx$

$= \frac{1}{2}x^4 e^{x^2} - \left[x^2 e^{x^2} - \int e^{x^2}(2x dx) \right]$

$= \frac{1}{2}x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + C$

$= \boxed{\frac{1}{2}e^{x^2}(x^4 - 2x^2 + 2) + C}$

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$